

QUIVER VARIETIES AND LUSZTIG'S ALGEBRA

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ABSTRACT. We study preprojective algebras of graphs and their relationship to module categories over representations of quantum $SL(2)$. As an application, ADE quiver varieties of Nakajima are shown to be subvarieties of the variety of representations of a certain associative algebra introduced by Lusztig.

1. INTRODUCTION

1.1. Motivated in part by G. Lusztig's work on quantum groups, H. Nakajima introduced in [21] remarkable algebraic varieties which we call Nakajima's quiver varieties. In [14–16] Lusztig produced another description of these varieties as certain Grassmannians. The key element of Lusztig's construction is an associative algebra attached to the quiver (this is the Lusztig's algebra from the title) and a map ϑ from Nakajima's varieties to the variety of representations of this algebra. In the ADE case the map ϑ is proved to be finite and a homeomorphism onto its image [15, Theorem 4.7].

The main purpose of the present paper is to prove (cf. Theorem 5.2) that if the quiver is of ADE type then the map ϑ is an algebraic isomorphism onto its image. This result is known in type A (cf. [18]) and was used in [20] to construct a compactification of Nakajima's varieties of type A. We plan to use techniques developed in the present paper to compactify Nakajima's varieties of arbitrary type.

1.2. Although our motivation comes from the geometry of quiver varieties the main technical results of the present paper are purely algebraic and concern the preprojective algebra of a graph. Let us recall the definitions. Consider a finite graph G with the vertex set I and the edge set H . We are mostly interested in graphs of finite and affine types, in other words in simply laced connected Dynkin or extended Dynkin diagrams, but at this point no restrictions on G are imposed. Now let us fix a field k and a vector $\lambda \in k^I$ and consider the following three k -algebras associated to G (these will be our main objects of study):

- the path algebra kG which is the k -linear space spanned by the set of all paths in G with multiplication (denoted by \cdot)

given by concatenation of paths. We consider λ as an element of kG under the natural identification of the subalgebra of trivial paths with k^I .

- the deformed preprojective algebra Π^λ which is the quotient of kG with respect to the two-sided ideal generated by the element $(\theta - \lambda)$, where

$$(1.2.a) \quad \theta = \sum_{\substack{i \in I \\ a \in H}} \theta_i^a.$$

Here $\theta_i^a \in kG$ is defined as follows. If an edge a has i as an end-vertex then θ_i^a is the path of length two from i to i along the edge a and back along the same edge, otherwise θ_i^a is zero.

- the Lusztig's algebra L^λ which is isomorphic to kG as a k -linear space, but with the following multiplication (cf. [16, 2.1]):

$$(1.2.b) \quad f \circ f' = f \cdot (\theta - \lambda) \cdot f',$$

where θ is as in (1.2.a).

The structure of Π^λ and L^λ depends on the choice of λ . For example Crawley-Boevey and Holland have shown (see [6, Corollary 6.3]) that if G is of finite type and λ does not lie on a Weyl chamber wall under the identification of k^I with the k -span of the weight lattice then $\Pi^\lambda = 0$ (zero algebra). Hence the most interesting case is $\lambda = 0$ when we have the original preprojective algebra Π^0 . It is also explained in [6] that in the affine case Π^λ is related to the algebra of global functions on a non-commutative deformation of a Kleinian singularity (see below for a more precise statement). The authors of [6] use slightly different definition of the preprojective algebra, but it is equivalent to ours for bipartite graphs (cf. the discussion in 1.5).

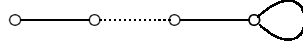
1.3. Our first theorem describes the Hilbert series of the preprojective algebra of an arbitrary graph. Let $\Pi^0(n)$ denote the n -th graded component of Π^0 with respect to the grading by path length. Given two vertices $i, j \in I$ let $\Pi_{ij}^0(n) = e_i \cdot \Pi^0(n) \cdot e_j$ be the subspace of paths going from j to i and put $H_{ij}(n) = \dim \Pi_{ij}^0(n)$. The matrix Hilbert series $H(t)$ of the preprojective algebra is defined as

$$H_{ij}(t) = \sum_{n=0}^{\infty} H_{ij}(n)t^n.$$

Theorem. *Assume $\text{char } k = 0$. Let G be a connected graph and A be the adjacency matrix of G .*

- (1.3.a) *If A is not of ADET type then $H(t) = (1 - At + t^2)^{-1}$. Moreover in this case the algebra Π^0 is Koszul.*
- (1.3.b) *If A is of ADET type then $H(t) = (1 + Pt^h)(1 - At + t^2)^{-1}$ where h is the Coxeter number of A and P is the permutation matrix corresponding to an involution of I .*

Here ADE refers to the simply laced Dynkin graphs and T_n is the following graph on n vertices:



The Coxeter number of T_n is $2n + 1$.

Part (a) of the Theorem is known in many cases, see [19]. However our approach is completely different and we hope that it would have other applications. Our observation is that one can think of the theory of preprojective algebras as of the invariant theory of the “quantum plane” with respect to the action of a “finite subgroup of quantum $SL(2)$ ”.

Let us recall that McKay correspondence relates extended Dynkin graphs and finite subgroups of $SL(2)$ and provides a very powerful approach to the study of affine quivers (see e.g. [6, 13, 14]). Now one can replace $SL(2)$ with quantum $SL_q(2)$ and study module categories over the category of representations of $SL_q(2)$, the module categories playing the role of subgroups in the quantum world. It turns out that this quantum version of the McKay correspondence allows one to obtain arbitrary graphs (cf. [9]). In the present paper we construct the preprojective algebra using the quantum McKay correspondence and give a simple proof of the above Theorem by reducing it to some standard facts about the quantum plane. In particular the special case of ADET graphs correspond to the deformation parameter q being a root of unity.

1.4. Our second theorem describes the zeroth Hochschild homology of the deformed preprojective algebra in the finite and affine cases.

Let us recall that a prime number p is called a *bad prime* for a simply laced Dynkin graph G if

$$\begin{aligned} p = 2 & \quad \text{for } G \text{ of type } D_n, \\ p = 2 \text{ or } 3 & \quad \text{for } G \text{ of type } E_6 \text{ or } E_7, \\ p = 2, 3, \text{ or } 5 & \quad \text{for } G \text{ of type } E_8. \end{aligned}$$

There are no bad primes for type A . If G is of affine type we say that p is bad if it is bad for the corresponding non-extended Dynkin graph. A prime which is not bad is *good*. Good and bad primes play

an important role in geometry and representation theory of algebraic groups. Their appearance in the following theorem is a mystery for us.

Theorem. *Let B be the image of the space of paths of length zero under the canonical projection $kG \rightarrow \Pi^\lambda$.*

(1.4.a) *Assume G is of finite ADE type and $\text{char } k$ is good for G . Then $\Pi^\lambda = [\Pi^\lambda, \Pi^\lambda] + B$.*

(1.4.b) *Assume G is of affine $\widehat{\text{ADE}}$ -type and $\text{char } k$ is good for G . Then $\Pi^\lambda = [\Pi^\lambda, \Pi^\lambda] + \Pi_{i_0 i_0}^\lambda + B$, where $\Pi_{i_0 i_0}^\lambda = e_{i_0} \cdot \Pi^\lambda \cdot e_{i_0}$ is the subalgebra of Π^λ consisting of all paths beginning and ending at a chosen extending vertex $i_0 \in I$.*

Recall that if R is an associative k -algebra then $[R, R]$ denotes the k -span of the set $\{ab - ba \mid a, b \in R\}$ and that a vertex of an affine Dynkin graph is called extending if by removing it one obtains a connected Dynkin graph of finite type.

The algebra $\Pi_{i_0 i_0}^\lambda$ which appears in the affine part of the above theorem is very important. As explained in [6] it should be thought of as the algebra of global functions on a non-commutative deformation of a Kleinian singularity. If $\text{char } k = 0$ statement 1.4.b can also be derived using McKay correspondence (cf. [5, 8.6]).

1.5. When $\lambda = 0$ the definition of the preprojective algebra contained in 1.2 is the original one given by I. M. Gelfand and V. A. Ponomarev and generalized by V. Dlab and C. M. Ringel. There is another definition in the literature which uses quivers (oriented graphs) instead of non-oriented graphs. Let us describe that construction. We fix an orientation Ω of G and denote by Q the resultant quiver. We continue to denote by H the set of edges (but now each edge is, in fact, an arrow). Let \overline{Q} be the double of Q , i.e. for each arrow $a \in H$ there is an extra arrow a^* of \overline{Q} connecting the same pair of vertices but going in the opposite direction.

Let $k\overline{Q}$ be the path algebra of the oriented graph \overline{Q} (if the graph G has no self-loops then it is canonically isomorphic to the path algebra kG) and let θ_Ω be the following element of $k\overline{Q}$ (cf. (1.2.a)):

$$(1.5.a) \quad \theta_\Omega = \sum_{a \in H} (a \cdot a^* - a^* \cdot a).$$

Then the deformed preprojective algebra Π_Ω^λ of the quiver Q is the quotient of $k\overline{Q}$ with respect to the two-sided ideal generated by the element $(\theta_\Omega - \lambda)$ and Lusztig's algebra L_Ω^λ is defined to be the vector space $k\overline{Q}$ equipped with the multiplication

$$f \circ f' = f \cdot (\theta_\Omega - \lambda) \cdot f'.$$

It is easy to see that the two definitions of the preprojective algebra are equivalent (i.e. Π^λ and Π_Ω^λ are isomorphic) for bipartite graphs (in particular for all finite and affine graphs except \hat{A}_{2n}). For arbitrary graphs the algebras are different but our results remain valid. Namely, the Hilbert series of Π_Ω^λ is given by Theorem 1.3 (with A being the adjacency matrix of \overline{Q}) and its zeroth Hochschild homology is described by Theorem 1.4. We will indicate the required modifications in the proofs.

The crucial difference between non-oriented graphs and doubled quivers is not the orientation and the related signs in the element θ but rather the doubling of self-loops, which changes dramatically the structure of the preprojective algebra. For example, if G is of type T_1 (one vertex, one edge graph) then $\Pi^0 \simeq k[x]/(x^2)$ while $\Pi_\Omega^0 \simeq k[x, y]$. Numerically each self-loop of G contributes 1 to the corresponding diagonal coefficient of the adjacency matrix A (which appears in Theorem 1.3) but it adds 2 to the coefficient of the adjacency matrix of \overline{Q} . Hence type T_1 is finite for G (i.e. $(2I - A)$ is positive-definite) but it is affine for \overline{Q} (i.e. $(2I - A)$ is indefinite). The Hilbert series of Π_Ω^0 is given by the expression in (1.3.a) if G is of ADE -type, and by (1.3.b) if it is not.

We decided to use the definition of the preprojective algebra given in 1.2 because it works more naturally in the context of the quantum McKay correspondence (particularly if the graph has self-loops). However both Π^λ and Π_Ω^λ can be obtained using module categories and we will indicate the required adjustments.

1.6. The paper is organized as follows. In Section 2 we describe a construction of the preprojective algebra via quantum McKay correspondence and prove Theorem 1.3. An important example of star-shaped graphs is discussed in Section 3. Section 4 contains the proof of Theorem 1.4. Finally in Section 5 we prove that Lusztig's map ϑ (cf. 1.1) is an isomorphism onto its image and describe applications to geometry and Poisson structure of quiver varieties.

1.7. Recently another construction of quiver varieties appeared in [10]. It uses so-called *skew-zigzag* algebra quadratic dual to the preprojective algebra. It would be very interesting to understand precisely the relationship (perhaps provided by Koszul duality) of the construction of [10] to Lusztig's construction and the present paper.

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2. QUANTUM MCKAY CORRESPONDENCE AND THE PREPROJECTIVE ALGEBRA

In this section we assume that k is an algebraically closed field of characteristic zero.

2.1. Quantum McKay Correspondence. As explained in the Introduction our goal is to obtain (and to exploit) a construction of the preprojective algebra of an arbitrary graph via quantum McKay correspondence. The original McKay observation was that given a finite subgroup $\Gamma \neq \{\pm 1\}$ of $SL(2)$ one can associate to it a simply laced affine Dynkin graph as follows. Let V be the fundamental 2-dim representation of $SL(2)$. Then the McKay graph has simple representations of Γ as vertices and two representations ρ and ρ' are connected by $\dim \operatorname{Hom}_\Gamma(\rho', \rho \otimes V)$ edges. McKay correspondence allows one to simplify many constructions and proofs in the particular case of an affine graph. For example a construction of the preprojective algebra of an affine quiver in terms of the invariant theory of Γ is given in [6, Section 3] and [14, Section 6]. Let us note that McKay theory relies only on the following facts:

- finite dimensional representations of $SL(2)$ form a semisimple tensor category \mathcal{C} ;
- representations of Γ form a semisimple module category \mathcal{D} over \mathcal{C} (i.e. there is a bifunctor $\mathcal{C} \times \mathcal{D} \rightarrow \mathcal{D}$ satisfying natural axioms, cf. [22, Section 2.3]);
- there is a self-dual object $V \in \mathcal{C}$ (the fundamental representation) such that the composition

$$(2.1.a) \quad \mathbf{1} \xrightarrow{\operatorname{coev}_V} V \otimes V^* \xrightarrow{\phi \otimes \phi^{-1}} V^* \otimes V \xrightarrow{\operatorname{ev}_V} \mathbf{1}$$

is equal to $-2 \operatorname{id}_\mathbf{1}$ for any choice of an isomorphism $\phi : V \rightarrow V^*$.

Following [9] we generalize this setup as follows. Let $q \in k \setminus \{0\}$ and consider the following tensor category \mathcal{C}_q : for $q = \pm 1$ or q not equal

to a root of unity \mathcal{C}_q is the category of representations of the quantum group $SL_q(2)$ while for q equal to a root of unity \mathcal{C}_q is the semisimple subquotient of the category of representations of $SL_q(2)$.

Let us recall the structure of \mathcal{C}_q (cf. [11]). First assume that q is not a root of unity. Then for each $s \in \mathbb{N}$ there is a simple object L_s of \mathcal{C}_q (the deformation of the $(s+1)$ -dimensional representation of $SL(2)$), the set $\{L_s\}_{s=0}^\infty$ is the complete set of simple objects, and the tensor product decomposition is given by

$$(2.1.b) \quad L_r \otimes L_s \simeq \bigoplus_{\substack{t=|r-s| \\ t \equiv r+s \pmod{2}}}^{r+s} L_t .$$

If q is a root of unity we put

$$(2.1.c) \quad h(q) = \begin{cases} N & \text{if } q \text{ is a root of unity of an odd order } N, \\ \frac{N}{2} & \text{if } q \text{ is a root of unity of an even order } N. \end{cases}$$

Then for each integer $s = 0, \dots, h(q) - 2$ there is a simple object L_s of \mathcal{C}_q (the deformation of the $(s+1)$ -dimensional representation of $SL(2)$), the set $\{L_s\}_{s=0}^{h(q)-2}$ is the complete set of simple objects, and the tensor product decomposition is given by

$$(2.1.d) \quad L_r \otimes L_s \simeq \begin{cases} \bigoplus_{\substack{t=|r-s| \\ t \equiv r+s \pmod{2}}}^{r+s} L_t & \text{if } r+s < h(q) - 1 ; \\ \bigoplus_{\substack{t=|r-s| \\ t \equiv r+s \pmod{2}}}^{2h(q)-4-r-s} L_t & \text{if } r+s \geq h(q) - 1 . \end{cases}$$

Let $V = L_1$ be the fundamental representation of $SL_q(2)$. The main result of [9] is a classification of semisimple module categories over \mathcal{C}_q with finitely many isomorphism classes of simple objects. Let \mathcal{D} be such a category and I be the set of isomorphism classes of simple objects of \mathcal{D} . Then as an abelian category \mathcal{D} is equivalent to the category of I -graded vector spaces which we denote by \mathcal{M}_I . Let $\text{Fun}(\mathcal{M}_I, \mathcal{M}_I) \cong \mathcal{M}_{I \times I}$ be the category of additive functors from \mathcal{M}_I to itself. The category $\text{Fun}(\mathcal{M}_I, \mathcal{M}_I)$ has an obvious structure of monoidal category induced by the composition of functors and the structure of a \mathcal{C} -module category on \mathcal{M}_I is just a tensor functor $F : \mathcal{C}_q \rightarrow \text{Fun}(\mathcal{M}_I, \mathcal{M}_I)$. According to [9], if q is not a root of unity then such functors are classified by the following data:

- a collection of finite dimensional vector spaces $\{V_{ij}\}_{i,j \in I}$;
- a collection of nondegenerate bilinear forms $E_{ij} : V_{ij} \otimes V_{ji} \rightarrow k$ satisfying the following conditions

$$(2.1.e) \quad \sum_{j \in I} \text{Tr}(E_{ij}(E_{ji}^t)^{-1}) = -q - q^{-1}$$

for each $i \in I$.

Here $\bigoplus_{i,j \in I} V_{ij} = F(V)$ is the object of $\mathcal{M}_{I \times I} \cong \text{Fun}(\mathcal{M}_I, \mathcal{M}_I)$ corresponding to $V \in \mathcal{C}_q$ and the forms E_{ij} are induced by an isomorphism $\phi : V \rightarrow V^*$. If q is a root of unity there is an extra condition due to the fact that \mathcal{C}_q is a quotient of the full tensor category generated by V . To avoid technical details we don't spell out the extra condition here and refer the reader to [9] instead.

Note that since E_{ij} is nondegenerate we have $\dim V_{ij} = \dim V_{ji}$. Let A be the symmetric $I \times I$ matrix given by $A_{ij} = \dim V_{ij}$. As in the original McKay correspondence we think of A as the adjacency matrix of a graph with vertex set I . Let us note that graphs equipped with a collection of bilinear forms assigned to edges are called modulated graph and they have been studied before in connection with non simply laced root systems (cf. [7]). So the quantum McKay correspondence produces modulated graphs satisfying condition (2.1.e).

The crucial fact is that by choosing an appropriate value of q and an appropriate module category \mathcal{D} we can obtain an arbitrary connected graph in this way. More precisely, according to [9], given a connected graph (i.e. dimensions of V_{ij} 's) the equation (2.1.e) has a solution (not necessarily unique) for some choice of q (in other words there exists a modulation of the graph satisfying (2.1.e)). We shall describe a solution to (2.1.e) in the course of the proof of Lemma 2.2.

It is shown in [9] that q is a root of unity if and only if the McKay graph is a Dynkin graph of ADDET type with Coxeter number $h(q)$ (cf. (2.1.c)).

2.2. Symmetric algebra. Let T be the free algebra in the category \mathcal{C}_q generated by V . Under the functor $F : \mathcal{C}_q \rightarrow \mathcal{M}_{I \times I}$ it maps to the path algebra of the McKay graph, i.e. to the algebra generated by the set of orthogonal idempotents $\{e_i\}_{i \in I}$ and the space of generators of degree one $\bigoplus_{i,j \in I} V_{ij}$.

Now consider the quotient S of T with respect to the two sided ideal generated by the image of $\mathbf{1}$ under the map $\mathbf{1} \xrightarrow{\text{coev}_V} V \otimes V^* \xrightarrow{\text{id}_V \otimes \phi^{-1}} V \otimes V$. If $q = 1$ (i.e. in the classical situation) S is just the algebra of polynomials in two commuting variables. In general S is called the q -symmetric algebra or the algebra of functions on the quantum

plane. Applying the functor F one gets an algebra $\tilde{\Pi}^E = F(S)$ which is the quotient of the path algebra with respect to the two sided ideal generated by $F(\mathbf{1} \subset V \otimes V)$. By the definition of F we have

$$F(\mathbf{1} \subset V \otimes V) = \sum_{i \in I} \theta_i^E,$$

where θ_i^E is defined as follows. Fix a pair of bases $x_1, \dots, x_{A_{ij}}$ in V_{ij} and $y_1, \dots, y_{A_{ij}}$ in V_{ji} dual with respect to the form E_{ji} and let $\theta_{ij}^E = \sum_{s=1}^{A_{ij}} y_s \cdot x_s$. Then the element θ_{ij}^E does not depend on the choice of the dual bases and we put $\theta_i^E = \sum_{j \in I} \theta_{ij}^E$. The algebra $\tilde{\Pi}^E$ is called the preprojective algebra of the modulated graph (cf. [8]).

Lemma. *Let G be an arbitrary connected graph and Π^0 be its preprojective algebra defined in 1.2. Then there exists a value of q and a \mathcal{C}_q -module category (in other words, a modulation $\{E_{ij}\}$ of G) such that $\Pi^0 = \tilde{\Pi}^E$.*

The lemma means that the preprojective algebra of an arbitrary connected graph can be realized via quantum McKay correspondence, or in other words, that the algebra of functions on the quantum plane is the “universal preprojective algebra”.

Proof. Let A be the adjacency matrix of G . It is irreducible since G is connected. Let λ be the Frobenius-Perron eigenvalue of A with eigenvector $\{r_i\}_{i \in I}$ (in particular $r_i \neq 0$ for any $i \in I$) and choose q so that $\lambda = -q - q^{-1}$. Now we can define the module category. Given two vertices $i, j \in I$ let V_{ij} be the linear space generated by the set of edges between i and j and define the bilinear forms $E_{ij} : V_{ij} \otimes V_{ji} \rightarrow k$ as follows:

$$(2.2.a) \quad E_{ij}(a, b) = \begin{cases} r_j & \text{if } a = b, \\ 0 & \text{if } a \neq b, \end{cases}$$

where a and b are two edges between i and j considered as basic elements of V_{ij} and V_{ji} respectively.

We claim that the forms E_{ij} satisfy the condition (2.1.e). Indeed,

$$\begin{aligned} \sum_{j \in I} \text{Tr}(E_{ij}(E_{ji}^t)^{-1}) &= \sum_{j \in I} \dim V_{ij} \left(\frac{r_j}{r_i} \right) = \\ &= \sum_{j \in I} A_{ij} \frac{r_j}{r_i} = \lambda \frac{r_i}{r_i} = -q - q^{-1} \end{aligned}$$

since $\{r_i\}_{i \in I}$ is an eigenvector of A with eigenvalue λ .

Now with this choice of bilinear forms we have

$$\theta_i^E = \frac{1}{r_i} \sum_{a \in H} \theta_i^a,$$

where θ_i^a is as in (1.2.a). It follows that $\tilde{\Pi}^E = \Pi^0$. \square

2.3. Hilbert Series. The prescription $\deg e_i = 0$ and $\deg x = 1$ for $x \in V_{ij}$ defines a grading on the algebra $\tilde{\Pi}^E$. Let $\tilde{\Pi}^E(n)$ denote the n -th graded component. Given $i, j \in I$ let $\tilde{\Pi}_{ij}^E(n) = e_i \tilde{\Pi}^E(n) e_j$ and let $H_{ij}(n) = \dim \tilde{\Pi}_{ij}^E(n)$. In this subsection we calculate the matrix Hilbert series $H(t) = \sum_{n=0}^{\infty} H_{ij}(n) t^n$ in terms of the adjacency matrix A .

Theorem. *Let $H(t)$ be the matrix Hilbert series of $\tilde{\Pi}^E$ defined above.*

(2.3.a) *Assume that the adjacency matrix A is not of ADET type. Then $H(t) = (1 - At + t^2)^{-1}$. Moreover in this case the algebra $\tilde{\Pi}^E$ is Koszul.*

(2.3.b) *Assume that the adjacency matrix A is of ADET type. Then $H(t) = (1 + Pt^h)(1 - At + t^2)^{-1}$, where h is the Coxeter number of A and P is the permutation matrix corresponding to some involution of I .*

Thanks to Lemma 2.2 the above Theorem is a generalization of Theorem 1.3 and in particular implies the latter. A minor technical point is that in Theorem 1.3 we don't assume k to be algebraically closed. However it is clear that $H(t)$ does not change under base field extensions.

Remark. It is easy to show that the Hilbert series of the quadratic dual of $\tilde{\Pi}^E$ is equal to $1 + At + t^2$ (except for several trivial graphs). It is also known that the Hilbert polynomial of the Yoneda algebra of $\tilde{\Pi}^E$ is given by $1 + At + t^2$ (for example it follows from the last part of the proof below). Therefore the algebra $\tilde{\Pi}^E$ is Koszul if and only if its Hilbert series is given by $(1 - At + t^2)^{-1}$.

Proof. The structure of the q -symmetric algebra S is well known, see [11]. In the case (2.3.a) (so q is not a root of unity) we have $S(n) = L_n$ for any $n \in \mathbb{N}$, where $S(n)$ is the n -th graded component of S . Now it follows from (2.1.b) that

$$L_1 \otimes L_n \simeq L_{n-1} \oplus L_{n+1} \quad \text{for } n > 0,$$

and applying the functor F we get a recursion

$$tAH(t) = H(t) + t^2H(t),$$

which implies the formula in (2.3.a).

In the case (2.3.b) (so q is a root of 1) we have $S(n) = L_n$ for $0 \leq n \leq h-2$ and $S(n) = 0$ for $n \geq h-1$. Let

$$\hat{S} = S \oplus t^h(L_{h-2} \otimes S) \oplus t^{2h}(L_{h-2} \otimes L_{h-2} \otimes S) \oplus \dots,$$

where powers of t denote grading shifts. Then (2.1.d) implies the following recursion

$$L_1 \otimes \hat{S}(n) \simeq \hat{S}(n-1) \oplus \hat{S}(n+1) \quad \text{for } n > 0,$$

and applying the functor F we obtain the formula in (2.3.b) with $P = F(L_{h-2})$. Note that $L_{h-2} \otimes L_{h-2} \simeq L_0 = \mathbf{1}$, hence $P^2 = \text{id}$.

Now assume that q is not a root of unity or $q = \pm 1$. Then the trivial left module $\mathbf{1} = L_0$ over the algebra S has the following resolution:

$$\mathbf{1} \rightarrow S \rightarrow S \otimes L_1 \rightarrow S \rightarrow \mathbf{0}$$

(this is just the q -version of the usual resolution for the algebra of polynomials in two commuting variables). The image of this resolution under the functor F provides a Koszul resolution for the algebra $\tilde{\Pi}^E$. \square

Remark. The involution P in (2.3.b) is an automorphism of the underlying graph and is explicitly known in all cases: for types A_n, D_{2n+1}, E_6 it is the unique nontrivial involution while for D_{2n}, T_n, E_7, E_8 it is the identity map.

2.4. In the *ADET* case the preprojective algebra is finite dimensional and we have the following result.

Corollary. *If the graph G is of ADET-type then*

$$\dim \Pi^0 = \frac{h(h+1)r}{6}.$$

where h and r are the Coxeter number and the rank respectively.

Proof. One has to put $t = 1$ in (2.3.b) and then calculate the sum of the matrix elements of the resultant matrix. In other words $\dim \Pi^0 = \text{Tr}(H(1)U)$, where the matrix U is given by $U_{ij} = 1$ for all $i, j \in I$. Observe that $PU = U$ and hence $\dim \Pi^0 = 2 \text{Tr}((2 - A)^{-1}U)$. Now assume we are in *ADE* situation. Then $(2 - A)$ is the Cartan matrix of the corresponding root system and hence $(2 - A)_{ij}^{-1} = (\omega_i, \omega_j)$ where ω_i are the fundamental weights and the scalar product is normalized by the condition $(\alpha, \alpha) = 2$ for a root α . Therefore $\dim \Pi^0 = 2 \sum_{i,j \in I} (\omega_i, \omega_j) = 2(\rho, \rho)$, where $\rho = \sum_{i \in I} \omega_i$. Now the result follows from the “strange formula” of Freudenthal and de Vries.

If G is of type T_n one can think of it as of a “folded” graph \tilde{G} of type A_{2n} . Then we have the natural surjective map of path algebras

$\sigma : k\tilde{G} \rightarrow kG$. The map σ respects the grading and the relations 1.2.a, and it is injective on the space of paths beginning at a fixed vertex of \tilde{G} . Since each vertex of G has two preimages in \tilde{G} we have $\dim \Pi^0(\tilde{G}) = 2 \dim \Pi^0(G)$ which implies the Corollary (recall that the Coxeter numbers of G and \tilde{G} are the same while the rank of \tilde{G} is twice the rank of G). \square

2.5. Quiver algebras. As discussed in Section 1.5, Theorem 1.3 remains valid for the preprojective algebra Π_Ω^0 defined using a double quiver \overline{Q} instead of a graph G (with A being the adjacency matrix of \overline{Q}). In fact one can construct the algebra Π_Ω^0 using quantum McKay correspondence. Namely, the following variant of Lemma 2.2 holds true and implies the analogue of Theorem 1.3.

Lemma. *Let Q be an arbitrary connected quiver. Then there exists a value of q and a \mathcal{C}_q -module category such that $\Pi_\Omega^0 = \tilde{\Pi}^E$.*

Proof. The proof is similar to the proof of Lemma 2.2. To define the module category we put V_{ij} to be the linear space generated by arrows of \overline{Q} going from i to j and define the bilinear forms E_{ij} as follows (recall that H is the set of arrows of Q):

$$E_{ij}(a, b) = \begin{cases} r_j & \text{if } a \in H \text{ and } b = a^* , \\ -r_j & \text{if } b \in H \text{ and } a = b^* , \\ 0 & \text{otherwise .} \end{cases}$$

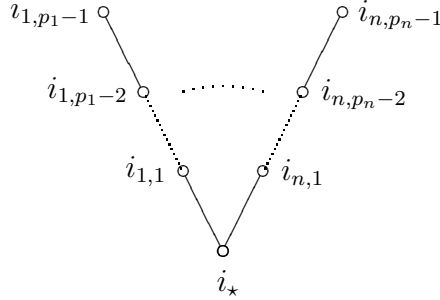
These data define a \mathcal{C}_q -module category provided $\{r_i\}_{i \in I}$ is the Frobenius-Perron eigenvector of the adjacency matrix A with the eigenvalue $\lambda = q + q^{-1}$. Moreover one has $\Pi_\Omega^0 = \tilde{\Pi}^E$. The calculations are similar to the ones in the proof of Lemma 2.2. \square

If A is affine then $\lambda = 2$ and $q = 1$. Thus we are in the classical McKay situation. In this case the above construction reduces to McKay description of the preprojective algebra given in [6, Section 3] and [14, Section 6]. Our choice of the bilinear forms E_{ij} is exactly the same as in *loc. cit.* (where r_i is the dimension of the irreducible representation ρ_i of Γ).

3. STAR GRAPHS

In this section we study star-shaped graphs. We assume $\text{char } k = 0$.

3.1. Spherical subalgebra. A star graph looks as follows:



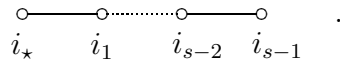
It has n rays of lengths p_1, \dots, p_n attached to the central vertex i_* . Graphs of this shape are important for two reasons. First they include all finite ADE types as well as some affine ones (\hat{D}_4 , \hat{E}_6 , \hat{E}_7 , and \hat{E}_8). Second they play an important role in classical linear algebra, such as various forms of Deligne-Simpson problem, saturation conjecture for Littlewood-Richardson coefficients, etc.. We would like to state this connection to matrix problems in the form of the following Proposition concerning the spherical subalgebra $\Pi_{i_*i_*}^0 = e_{i_*} \cdot \Pi^0 \cdot e_{i_*}$ of the preprojective algebra Π^0 consisting of paths beginning and ending at the central vertex.

Proposition. *The unital algebra $\Pi_{i_*i_*}^0$ has the following presentation over k :*

$$\Pi_{i_*i_*}^0 = \langle x_r, \quad r = 1, \dots, n \mid x_1^{p_1} = \dots = x_n^{p_n} = x_1 + \dots + x_n = 0 \rangle.$$

*The presentation is given by sending x_r to the loop of length one along the r -th ray of the graph (i.e. $x_r = [i_*i_{r,1}i_*]$).*

Proof. Let R_s be the graph



One should think of R_s as of a ray of the star graph. Consider the algebra Φ_s defined as

$$\Phi_s = kR_s / \left(\sum_{m=1}^{s-1} \theta_{i_m} \right),$$

where θ_{i_m} is as in (1.2.a). The algebra Φ_s is almost the preprojective algebra of the graph R_s except we don't impose the relation θ_{i_*} . Let $\Psi_s = e_{i_*} \cdot \Phi_s \cdot e_{i_*}$ be the subalgebra of paths beginning and ending at the star vertex. An easy induction on s shows that

$$\Psi_s = k[x] / (x^s)$$

with $x = [i_* i_1 i_*]$. Now the Proposition follows from the fact that for the original star graph

$$\begin{aligned} \Pi_{i_* i_*}^0 &= \Psi_{p_1} * \dots * \Psi_{p_n} / (\theta_{i_*}) = \\ &= (k[x_1]/(x_1^{p_1})) * \dots * (k[x_n]/(x_n^{p_n})) / (x_1 + \dots + x_n), \end{aligned}$$

where $*$ denotes the free product of algebras. \square

3.2. Hilbert Series. We saw that the calculation of the Hilbert series of a preprojective algebra reduces to the inversion of the matrix $1 - At + t^2$. In the case of a star-shaped graph Lusztig and Tits [17] derived the following expression for the “central” coefficient of the inverse:

$$((1 - At + t^2)^{-1})_{i_* i_*} = \left(1 + t^2 - t \sum_{s=1}^n \frac{t^{p_s-1} - t^{-p_s+1}}{t^{p_s} - t^{-p_s}}\right)^{-1}.$$

Using Theorem 1.3 and the above formula we obtain the following result (note that $\deg x_r = 2$ in the standard path algebra grading).

Proposition. *Let $h(t)$ be the Hilbert series of the subalgebra $\Pi_{i_* i_*}^0$ with respect to the grading defined by $\deg(x_r) = 1$.*

(3.2.a) *If the star graph is not of ADE type then*

$$h(t) = \left(1 + t - \sum_{s=1}^n \frac{t - t^{p_s}}{1 - t^{p_s}}\right)^{-1}.$$

(3.2.b) *If the star graph is of DE type then*

$$h(t) = \left(1 + t - \sum_{s=1}^n \frac{t - t^{p_s}}{1 - t^{p_s}}\right)^{-1} (1 + t^{h/2}).$$

As an example, for E_8 we obtain

$$\begin{aligned} h(t) &= \left(1 + t - \frac{t - t^2}{1 - t^2} - \frac{t - t^3}{1 - t^3} - \frac{t - t^5}{1 - t^5}\right)^{-1} (1 + t^{15}) = \\ &= 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 6t^5 + 6t^6 + 6t^7 + 6t^8 + 6t^9 + 5t^{10} + 4t^{11} + 3t^{12} + 2t^{13} + t^{14} \end{aligned}$$

3.3. Kleinian groups. Finally we would like to explain the connection between star shaped graphs and the McKay correspondence. Let us recall that finite subgroups of $SL(2)$ are in one-to-one correspondence with classical Dynkin graphs of ADE type. Let a , b , and c , be the lengths of the rays of the graph corresponding to a subgroup $\Gamma \subset SL(2)$.

In the A_{2n-1} -case we put $a = b = n$ and $c = 1$ and we exclude A_{2n} -case to simplify discussion. Then $-I \in \Gamma$ and we have the following presentation (cf. [2]):

$$\Gamma / \{\pm I\} = \langle X, Y, Z \mid X^a = Y^b = Z^c = XYZ = 1 \rangle,$$

which is a multiplicative analogue of the presentation of the spherical subalgebra.

Proposition. *The algebra $\Pi_{i_* i_*}^0$ has the following presentation*

$$\Pi_{i_* i_*}^0 = \langle x, y, z \mid x^a = y^b = z^c = x + y + z = 0 \rangle.$$

Moreover $\dim \Pi_{i_* i_*}^0 = |\Gamma|/2$.

Proof. The first statement is a special case of Proposition 3.1. The dimension of $\Pi_{i_* i_*}^0$ in the DE case is given by 3.2.b with $t = 1$ and in the A_{2n-1} case can be easily found from the presentation. The answer is the same in both cases:

$$\dim \Pi_{i_* i_*}^0 = \frac{2}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1} = |\Gamma|/2.$$

□

Remark. As it was explained to us by P. Etingof one can both prove and generalize the equality $\dim \Pi_{i_* i_*}^0 = |\Gamma|/2$ using the holonomy of the connection $(\frac{x}{\xi} + \frac{y}{\xi+1} + \frac{z}{\xi-1})d\xi$ on $\mathbb{C} \setminus \{0, \pm 1\}$.

4. PROOF OF THEOREM 1.4

4.1. This section is devoted to the proof of Theorem 1.4 (the same argument works for preprojective algebras of double quivers – cf. 1.5). The proof consists of several reductions followed with case-by-case calculations. First note that since Π^λ is filtered by the path length it is enough to prove the Theorem for the associated graded algebra which is isomorphic to a quotient of Π^0 . So we assume $\lambda = 0$.

4.2. Next we do a reduction on the support of (a class of) a path. Let i be a vertex of G , $\Pi_{ii}^0 = e_i \cdot \Pi^0 \cdot e_i$ be the subalgebra of Π^0 consisting of classes of paths beginning and ending at i , $\Pi_{G \setminus i}^0$ be the preprojective algebra of the graph obtained by removing the vertex i and the adjacent edges from G , and J_i be the two-sided ideal of Π^0 generated by e_i (so J_i consists of paths “passing through i ”). We claim that $J_i \subset \Pi_{ii}^0 + [\Pi^0, \Pi^0]$. Indeed any non-cyclic path belongs to $[\Pi^0, \Pi^0]$ and any cyclic path passing through i can be made to begin and end at i after a cyclic permutation (i.e. after adding an element of $[\Pi^0, \Pi^0]$). Now we have a natural surjective map of algebras $\Pi_{G \setminus i}^0 \rightarrow \Pi^0/J_i$ and

hence (1.4.b) follows from (1.4.a) if we put i to be an extending vertex while (1.4.a) can be proven by induction on the number of vertices once it is shown that $\Pi_{ii}^0 \subset [\Pi^0, \Pi^0] + B$ for some vertex i . The verification of this last inclusion is the following case-by-case calculation.

G is of type A_n . We choose i to be one of the two end-points of the Dynkin graph. Then according to (a degenerate version of) Proposition 3.1 we have

$$\Pi_{ii}^0 = \langle x \mid x^n = x = 0 \rangle = ke_i \subset B$$

G is of type D_n or E_n . We choose i to be an end-point of the Dynkin graph furthest from the three-valent point. Let $f \neq e_i$ be a path in G beginning and ending at i . We are going to prove that $f = 0$ modulo the commutant and the ideal 1.2.a. In what follows all the calculations are done modulo these subspaces. First of all we may assume that f passes through the three-valent vertex (otherwise we are in type A situation). Therefore (modulo the commutant) we assume that f begins and ends at the three-valent vertex and passes through the end-point. We will use the notation of Proposition 3.3 with $(a, b, c) = (n - 2, 2, 2)$ (resp. $(n - 3, 3, 2)$) in type D_n (resp. E_n). So f is a word in the alphabet $\{x, y, z\}$ containing x^{a-1} . We have to prove that $f = 0$ if

$$x^a = y^b = z^c = x + y + z = 0$$

and we are allowed to do cyclic permutations. In other words, f is a *cyclic* word in y and z subject to the following relations:

$$(4.2.a) \quad y^b = 0$$

$$(4.2.b) \quad z^c = 0$$

$$(4.2.c) \quad (y + z)^a = 0$$

and we want to show that $f = 0$ if $f = (y + z)^{a-1}g$ for some $g \neq 1$ (otherwise we are again in type A situation). Moreover because of (4.2.c), we can assume that the word g begins and ends with z .

4.3. D_n -case. We have

$$(4.3.a) \quad y^2 = 0$$

$$(4.3.b) \quad z^2 = 0$$

$$(4.3.c) \quad (y + z)^{n-2} = 0$$

which implies that

$$(4.3.d) \quad (y + z)^k = \underbrace{yzyz \dots y(z)}_k + \underbrace{zyzy \dots z(y)}_k$$

for any k . Now let $f = (y + z)^{n-3}g$ for some word $g \neq 1$ beginning with z . There are two cases to consider: $g = z$ and $g = zyh$ for some h ($h = 1$ is allowed). If $g = z$ equation (4.3.d) implies (recall that f is a cyclic word)

$$\begin{cases} f = 0 & \text{if } n \text{ is odd} \\ 2f = (y + z)^{n-2} = 0 & \text{if } n \text{ is even} \end{cases}$$

If $g = zyh$ we have

$$f = \underbrace{(y)zyz \dots yz}_{n-2}yh \stackrel{(4.3.c)}{=} - \underbrace{(z)yz y \dots zy}_{n-2}yh \stackrel{(4.3.a)}{=} 0,$$

which completes the proof for D_n assuming $\text{char } k \neq 2$.

4.4. E_6 -case. We have

$$(4.4.a) \quad y^3 = 0$$

$$(4.4.b) \quad z^2 = 0$$

$$(4.4.c) \quad (y + z)^3 = y^2z + yzy + zy^2 + zyz = 0$$

and $f = (y + z)^2g = y^2g$ for some word $g \neq 1$ beginning and ending with z . There are two cases: $g = z$ and $g = zhz$ for some word h beginning and ending with y . If $g = z$ we have (recall that f is a cyclic word)

$$3f = 3y^2z = (y + z)^3 \stackrel{(4.4.c)}{=} 0.$$

If $g = zhz$ we have

$$f = y^2zhz \stackrel{(4.4.c)}{=} -yzyhz = -zyzyh \stackrel{(4.4.c)}{=} (y^2z + yzy + zy^2)yh = 0$$

since h begins and ends with y and $y^3 = 0$. This completes the proof for E_6 assuming $\text{char } k \neq 2, 3$ (one needs the assumption $\text{char } k \neq 2$ because E_6 contains subgraphs of type D_n).

4.5. E_7 -case. We have

$$(4.5.a) \quad y^3 = 0$$

$$(4.5.b) \quad z^2 = 0$$

$$(4.5.c) \quad (y + z)^4 = y^2zy + yzy^2 + yzyz + zy^2z + zyzy = 0$$

and $f = (y + z)^3g = yzyg$ for some word $g \neq 1$ beginning and ending with z . There are two cases: $g = z$ and $g = zhz$ for some word h beginning and ending with y . If $g = z$ we have (recall that f is a cyclic word)

$$2f = 2yzyz \stackrel{(4.5.a, 4.5.b)}{=} (y + z)^4 \stackrel{(4.5.c)}{=} 0.$$

If $g = zhz$ we have

$$f = yzyzhz \stackrel{(4.5.c)}{=} -y^2zyhz = -zy^2zyh$$

since h begins and ends with y and $y^3 = 0$. So f is a cyclic word in syllabi zy and zy^2 having length greater than 5 and containing zy^2 . We are going to prove that any such word is equal to 0.

Using (4.5.c) we get the following relations:

$$(4.5.d) \quad zyzzyzy^2 = 0 \quad [z(4.5.c)y^2],$$

$$(4.5.e) \quad zy^2zyzy^2 + zyzzy^2zy^2 = 0 \quad [z(4.5.c)zy^2],$$

$$(4.5.f) \quad zy^2zy^2zy^2 = 0 \quad [zy^2(4.5.c)y^2].$$

If f contains only one zy^2 and has length greater than 5 it vanishes because of (4.5.d). If there are more than one zy^2 we can use (4.5.e) to move syllabi of the type zy together and then conclude from (4.5.d) that the cyclic word f vanishes if it contains more than one such syllabi. If f contains one zy then (4.5.e) implies that $2f = 0$. The relation (4.5.f) proves that any word having more than two syllabi zy^2 and having no syllabi zy vanishes, so the only remaining case is $f = zy^2zy^2$, which is equal to 0 because of (4.5.c, 4.5.a). This completes the proof for E_7 assuming $\text{char } k \neq 2, 3$.

4.6. E_8 -case. We have

$$(4.6.a) \quad y^3 = 0$$

$$(4.6.b) \quad z^2 = 0$$

$$(4.6.c) \quad (y+z)^5 = y^2zy^2 + y^2zyz + yzy^2z + yzyzy + zy^2zy + zyzzy^2 + zyzzyz = 0$$

and $f = (y+z)^4g = y^2zyg + yzy^2g$ for some word $g \neq 1$ beginning and ending with z . There are two cases: $g = z$ and $g = zhz$ for some word h beginning and ending with y . If $g = z$ we have (recall that f is a cyclic word)

$$5f = 10zyzy^2 \stackrel{(4.6.a, 4.6.b)}{=} 2(y+z)^5 \stackrel{(4.6.c)}{=} 0.$$

If $g = zhz$ we have

$$f = y^2zyzhz + yzy^2zhz = zy^2zyzh + zyzzy^2zh.$$

Hence f is a sum of cyclic words in syllabi zy and zy^2 having length greater than 6 and containing zy^2 . We are going to prove that any such word is equal to 0.

Using (4.6.c) we get the following relations:

$$\begin{aligned}
 (4.6.d) \quad & zy^2zyzy + zyzzy^2zy + zyzzyzy^2 = 0 \quad [z(4.6.c)y] , \\
 (4.6.e) \quad & zy^2zyzy^2 + zyzzy^2zy^2 = 0 \quad [z(4.6.c)y^2] , \\
 (4.6.f) \quad & zy^2zy^2zy^2 + zyzzyzyzy^2 = 0 \quad [z(4.6.c)zy^2] , \\
 (4.6.g) \quad & zy^2zyzyzy^2 = 0 \quad [zy^2(4.6.c)y^2] .
 \end{aligned}$$

Consider a cyclic word w in syllabi zy and zy^2 containing zy^2 and having length greater than 6. If w has only one zy^2 then (4.6.d) implies that $3w = 0$. If there are more than one zy^2 's we can first use (4.6.f) to remove strings of zy 's longer than $zyzy$. Now if there are two or more zy 's we can move them together using (4.6.e) and then conclude from (4.6.g) that the cyclic word w vanishes. If w contains one zy then (4.6.e) implies that $2w = 0$. So the remaining case is a cyclic word having only syllabi of the type zy^2 and, moreover, having three or more such syllabi (since the length is greater than 6). In this case we can use (4.6.f) to create a substring $zyzyzyzy^2$ inside the word w . Now if $w = zyzyzyzy^2$ then $3w = 0$ because of (4.6.d). Otherwise w contains the string $zy^2zyzyzyzy^2$ and so it vanishes because of (4.6.d, 4.6.g). This completes the proof for E_8 assuming $\text{char } k \neq 2, 3, 5$.

5. QUIVER VARIETIES

In this section the graph G is of ADE type and k is an algebraically closed field of characteristic zero.

5.1. Quivers. Since we plan to discuss, in particular, Poisson structures which are antisymmetric by nature we need to choose an orientation of G (i.e. make it into a quiver Q) and use the quiver-based algebras Π_Ω^λ and L_Ω^λ (cf. 1.5). The graph being Dynkin (and hence bipartite) it is easy to see that $L_\Omega^\lambda \simeq L^\lambda$ and $\Pi_\Omega^\lambda \simeq \Pi^\lambda$ as graded algebras.

5.2. Varieties. Now let us turn to geometry. Given I -graded vector spaces D and V , Nakajima [21] considers the affine algebraic variety (in fact a linear space) $N_{D,V}$ of all triples (x, p, q) , where x is a graded representation of the path algebra $k\overline{Q}$ in V , and $p : D \rightarrow V$ and $q : V \rightarrow D$ are graded linear maps. The variety $N_{D,V}$ is Poisson with respect to the bivector $\sum_{i \in I} \text{Tr}(\partial_{p_i} \wedge \partial_{q_i}) + \sum_{a \in H} \text{Tr}(\partial_{x(a)} \wedge \partial_{x(a^*)})$.

Let $GL(V)$ denote the group of graded automorphisms of V and $\mathfrak{gl}(V)$ be its Lie algebra. Then $GL(V)$ acts naturally on $N_{D,V}$ preserving the Poisson structure and with the moment map $\mu : N_{D,V} \rightarrow \mathfrak{gl}(V)^*$

given by

$$\mu(x, p, q) = \sum_{a \in H} (x(a)x(a^*) - x(a^*)x(a)) - \sum_{i \in I} p_i q_i = x(\theta_\Omega) - pq ,$$

where we consider $\lambda \in k^I$ as the diagonal matrix $\sum_{i \in I} \lambda_i \text{id}_{V_i} \in \mathfrak{gl}(V)$ and identify $\mathfrak{gl}(V)$ and $\mathfrak{gl}(V)^*$ via the trace form. Let $\mathfrak{N}_{D,V,\lambda}$ be the symplectic quotient at the value of the moment map equal to λ :

$$\begin{aligned} \mathfrak{N}_{D,V,\lambda} &= \mu^{-1}(\lambda) // GL(V) = \\ &= \{(x, p, q) \in N_{D,V} \mid x(\theta_\Omega - \lambda) = pq\} // GL(V) . \end{aligned}$$

Here $//$ denotes the affine quotient (spectrum of the ring of invariant global functions). The associated reduced scheme $\mathfrak{N}_{D,V,\lambda}^{\text{red}}$ was introduced by Nakajima [21] who also constructed a resolution of singularities of $\mathfrak{N}_{D,V,\lambda}^{\text{red}}$.

On the other hand, following Lusztig [16, 2.3], one can consider the affine scheme $\mathfrak{L}_{D,\lambda}$ of graded representations of the algebra L_Ω^λ in the I -graded vector space D (it is proven by Lusztig [16, Lemma 2.2] that L_Ω^0 is finitely generated and the proof works for arbitrary $\lambda \in k^I$). Note that $\mathfrak{L}_{D,\lambda}$ is the scheme of *all* representations not of isomorphism classes.

One has a natural map

$$\vartheta : \mathfrak{N}_{D,V,\lambda} \rightarrow \mathfrak{L}_{D,\lambda}$$

given by

$$\vartheta((x, p, q))(f) = qx(f)p ,$$

where $f \in k\overline{Q}$ is a path in \overline{Q} . The moment map condition $x(\theta_\Omega - \lambda) = pq$ implies that

$$\begin{aligned} \vartheta((x, p, q))(f)\vartheta((x, p, q))(f') &= \\ &= \vartheta((x, p, q))(f \cdot (\theta_\Omega - \lambda) \cdot f') = \vartheta((x, p, q))(f \circ f') . \end{aligned}$$

Hence the map ϑ is well-defined.

Lusztig proved that ϑ is injective and finite [15, Theorem 4.7] and that $\mathfrak{L}_{D,\lambda}$ is the union of images of maps ϑ for various V [15, Lemma 4.12]. The following Theorem is a refinement of the Lusztig's result.

Theorem. *The map ϑ is an immersion of Nakajima's scheme $\mathfrak{N}_{D,V,\lambda}$ as a subscheme of the scheme $\mathfrak{L}_{D,\lambda}$ of representations of L_Ω^λ in D .*

Proof. It is shown by Lusztig [14, Section 1] and it also follows from the result of Le Bruyn and Procesi [1, Section 3] using a trick of Crawley-Boevey [3, Remarks after the Introduction] that the algebra $k[\mathfrak{N}_{D,V,\lambda}]$ of global functions on $\mathfrak{N}_{D,V,\lambda}$ is generated by matrix elements of the

operators $qx(f)p : D \rightarrow D$ and traces of the operators $x(f) : V \rightarrow V$. The Theorem means that elements of the first type (which come from $\mathfrak{L}_{D,\lambda}$ via ϑ^{-1}) already generate $k[\mathfrak{N}_{D,V,\lambda}]$. Let us prove it. Consider $\text{Tr } x(f)$ for some path $f \in k\overline{Q}$. We can use Theorem 1.4 (more precisely, its variant for Π_Ω^λ , or the original Theorem together with an isomorphism $\Pi_\Omega^\lambda \simeq \Pi^\lambda$) to write f as a sum $f = f_\theta + f_{[\cdot]} + f_0$, where f_θ belongs to the two sided ideal generated by θ_Ω , $f_{[\cdot]} \in [k\overline{Q}, k\overline{Q}]$, and f_0 has length zero. Now $\text{Tr } x(f_0)$ is constant on $\mathfrak{N}_{D,V,\lambda}$ (it depends only on the graded dimension of V), $\text{Tr } x(f_{[\cdot]}) = 0$, and the moment map condition $x(\theta_\Omega) = pq$ implies that $\text{Tr } x(f_\theta) = \text{Tr } qx(f')p$ for some $f' \in k\overline{Q}$. \square

Remark. A similar statement in the affine case would say that the fibers of the map ϑ are related to the symmetric powers of Kleinian singularities. That statement is not made precise in the present paper (note in particular that $\mathfrak{L}_{D,\lambda}$ is infinite-dimensional in the affine case). Also Lusztig's algebra L_Ω^λ in the affine case can be described via McKay correspondence. Namely it is isomorphic as a vector space to $u(k\langle x, y \rangle \# k\Gamma)u$, where $k\langle x, y \rangle$ is the ring of noncommutative polynomials in two variables, Γ is a finite subgroup of $SL(2, k)$, and u is the sum of orthogonal idempotents in $k\Gamma$, but the multiplication in L_Ω^λ is given by the insertion of the element $(xy - yx - \lambda)$ with λ considered as an element of the center of $k\Gamma$ (cf. (1.2.b)).

5.3. Poisson structures. Recall the following facts about Nakajima's varieties $\mathfrak{N}_{D,V,\lambda}^{\text{red}}$ (cf. [16, 21]):

- $\{\mathfrak{N}_{D,V,\lambda}^{\text{red}}\}$ form an inductive system of Poisson varieties with respect to embeddings of the I -graded vector spaces V ;
- $\mathfrak{L}_{D,\lambda}^{\text{red}} \subset \bigcup_V \text{im}(\vartheta : \mathfrak{N}_{D,V,\lambda} \rightarrow \mathfrak{L}_{D,\lambda})$;
- $\mathfrak{N}_{D,V,\lambda}^{\text{red}}$ contains a smooth open (possibly empty) symplectic subset $\mathfrak{N}_{D,V,\lambda}^s$ (the set of stable points) such that the complement of $\mathfrak{N}_{D,V,\lambda}^s$ in $\mathfrak{N}_{D,V,\lambda}^{\text{red}}$ is the union of images of the natural embedding maps $\mathfrak{N}_{D,V',\lambda}^{\text{red}} \rightarrow \mathfrak{N}_{D,V,\lambda}^{\text{red}}$ for $V' \subset V$;
- there are finitely many non-isomorphic V 's such that $\mathfrak{N}_{D,V,\lambda}^s$ is non-empty.

Therefore Theorem 5.2 implies the following.

Corollary. *The variety $\mathfrak{L}_{D,\lambda}^{\text{red}}$ of representations of L_Ω^λ in D is Poisson. Moreover it is a union of finitely many symplectic leaves $\vartheta(\mathfrak{N}_{D,V,\lambda}^s)$; in particular, each symplectic leaf is the smooth locus of its closure.*

Remark. Instead of using the inductive system of varieties $\mathfrak{N}_{D,V,\lambda}^{\text{red}}$ to define the Poisson structure on the variety $\mathfrak{L}_{D,\lambda}^{\text{red}}$ of representations

of the algebra L_Ω^λ one can, in the spirit of non-commutative symplectic geometry (cf. [5, 12]), define a (non-commutative) Poisson bracket directly on L_Ω^λ as follows:

$$\{f', f''\} = \sum_{a \in H} (P_a(f', f'') - P_a(f'', f')) + f' \cdot f'' - f'' \cdot f',$$

where the path $P_a(f', f'')$ is obtained from the paths f' and f'' by locating an edge a in f' and an edge a^* in f'' (if there are any), removing these edges and gluing the resultant paths together and \cdot is, as before, the concatenation of paths. Then the Poisson structure descends to the variety $\mathfrak{L}_{D,\lambda}^{\text{red}}$ of representations of L_Ω^λ in any graded vector space D . This universal construction produces the same Poisson bracket as the one defined via the maps $\vartheta : \mathfrak{N}_{D,V,\lambda} \rightarrow \mathfrak{L}_{D,\lambda}$.

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